Modeling Interference with α-stable and Copulas

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Section 1

System Model

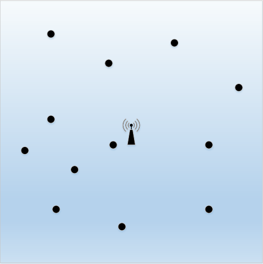


Figure 1: System Model

- Receiver at origin
- Interfering nodes: PPP with λ on the whole plane — Φ
  - The radius of the network $r_{max} \to \infty$
  - No guard zone $r_{min} \to 0$
  - $r_j \in (0, \infty)$
- Independent in time

Received interference (amplitude):

$$Z = \sum_{j \in \Phi} r_j^{-\eta/2} h_j x_j, \tag{1}$$

System Model

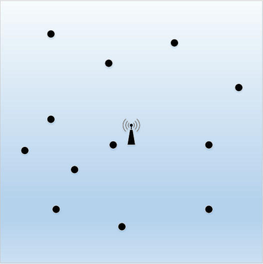


Figure 1: System Model

- PPP: Φ
- Path loss: η
- Base band emission: x_j
- Rayleigh fading: $h_j \sim \mathcal{CN}(0, 1)$

Received interference (amplitude):

$$Z = \sum_{j \in \Phi} r_j^{-\eta/2} h_j x_j, \tag{1}$$

Interference Model

Received interference:

$$Z = \sum_{j \in \Phi} r_j^{-\eta/2} h_j x_j = z_r + i \cdot z_i. \quad (2)$$

Theorem 1

If $h_j x_j$ satisfies

$$\mathbb{E}[|\mathrm{Re}(h_j x_j)|^{4/\eta}|] < \infty, \quad (3)$$

z_r is symmetric α-stable ($S\alpha S$) random variables with $\alpha = 4/\eta$,

Characteristic Function of $X \sim S_\alpha(\sigma, \beta, \mu)$:

$$\mathbb{E}[e^{i\theta X}] = \begin{cases} \exp\left\{-\gamma^\alpha |\theta|^\alpha (1 - i\beta(\mathrm{sign}\theta)\tan\frac{\pi\alpha}{2}) + \delta\theta\right\}, \alpha \neq 1 \\ \exp\left\{-\gamma|\theta|(1 + i\beta\frac{2}{\pi}(\mathrm{sign}\theta)\log|\theta|) + i\delta\theta\right\}, \alpha = 1 \end{cases} \quad (4)$$

Interference Model

Received interference

$$Z = z_r + i \cdot z_i. \quad (5)$$

Theorem 2

Suppose that $h_j x_j$ is an isotropic complex random variable and

$$\mathbb{E}[|\mathrm{Re}(h_j x_j)|^{4/\eta}|] < \infty, \quad (6)$$

$\mathbf{Z} = (z_r, z_i)$ *follows sub-Gaussian distribution with $\alpha = 4/\eta$,*

Definition 3

Any vector $\mathbf{X}$ distributed as $\mathbf{X} = (A^{1/2}G_1, \ldots, A^{1/2}G_d)$, where

$$A \sim S_{\alpha/2}((\cos \pi\alpha/4)^{2/\alpha}, 1, 0), \quad (7)$$

and $\mathbf{G} = [G_1, \ldots, G_d]^T \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ is called a sub-Gaussian α-stable random vector in $\mathbb{R}^d$ with underlying Gaussian vector $\mathbf{G}$.

Section 2

System Model

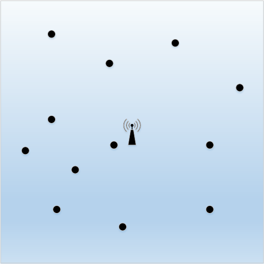
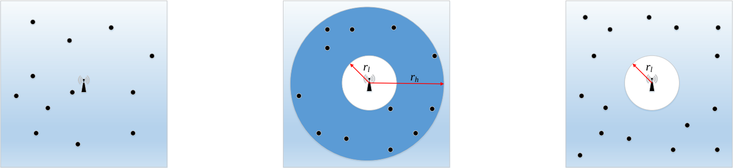


Figure 2: System Model

- Receiver at origin
- Interfering nodes: PPP with λ on the whole plane — Φ
  - The radius of the network $r_{max} \to \infty$
  - No guard zone $r_{min} \to 0$
  - $r_j \in (0, \infty)$
- Independent in time

The validity of α-stable model in more realistic conditions.

System Model



(a) Case 1, (b) Case 2, (c) Case 3,

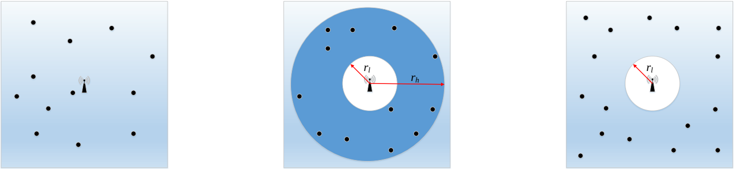
Figure 3: Systerm Model

Case 1 $r \in (0, \infty)$

Case 2 $r \in (r_l, r_m]$

Case 3 $r \in (r_l, \infty)$

System Model



(a) Case 1, (b) Case 2, (c) Case 3,

Figure 3: Systerm Model

Case 1 $r \in (0, \infty)$
Case 2 $r \in (r_l, r_m]$
Case 3 $r \in (r_l, \infty)$

Under which conditions of r_l, is the isotropic α-stable model a good approximation of the interference?

Numerical Results

Simulation Parameters

- $\lambda = 0.001$
- $r_{max} = 5000$

By varying guard zon radius – r_l, we evaluate the accuracy of α-stable marginals in terms of the estimated stable parameter – $\hat{\alpha}$ and quantiles (Q-Q plots).
We also evaluate the dependence strucutre of sub-Gaussian in copula space.

Numerical Results

1. Estimated stable parameter – $\hat{\alpha}$

We fitted the simulated data with the α-stable model, then we can get the correspoding estimated parameters $\hat{\alpha}$ which indicates the impulsiveness.

Table 1: Estimated α of Model II

r_I (m)	0.05	0.5	2.5	5.0	7.5	10.0
$\hat{\alpha}$	0.805	0.807	0.831	0.916	1.114	1.327
r_I (m)	12.5	15.0	17.5	20.0	22.5	25.0
$\hat{\alpha}$	1.532	1.663	1.755	1.825	1.87	1.894
r_I (m)	27.5	30.0	32.5	35.0	37.5	40.0
$\hat{\alpha}$	1.929	1.939	1.957	1.959	1.963	1.968
r_I (m)	42.5	45.0	47.5	50.0		
$\hat{\alpha}$	1.972	1.976	1.979	1.982		

Numerical Results

2. Quantiles

We use the empirical cumulative distribtued function (ECDF) of the simulated data from case 1 and case 3 to get the Q-Q plots under different r_I.

Q-Q Plot

A Q–Q plot is a plot of the quantiles of two distributions against each other, or a plot based on estimates of the quantiles. If the two distributions being compared are identical, the Q–Q plot follows the line $y = x$.

Numerical Results

Q_0 — the quantile from the Case 1, i.e., $r_I = 0$
Q_I — the quantile from the Case 3, i.e., $r_I > 0$

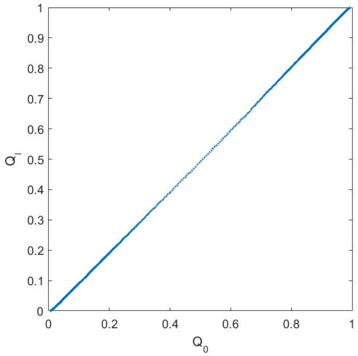


Figure 4: Q-Q plots with $r_I = 2.5m$

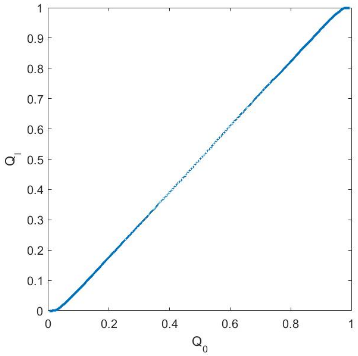


Figure 5: Q-Q plots with $r_I = 5m$

Numerical Results

Q_0 — the quantile from the Case 1, i.e., $r_I = 0$
Q_I — the quantile from the Case 3, i.e., $r_I > 0$

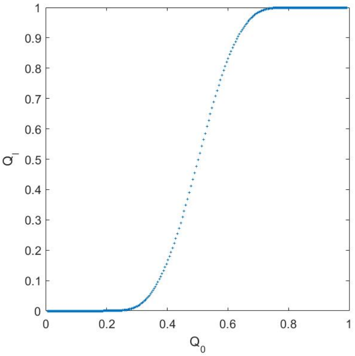


Figure 6: Q-Q plots with $r_I = 30m$

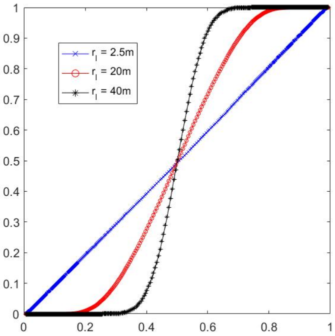


Figure 7: Q-Q plots with different r_I

Numerical Results

Using the empirical CDF, we map the simulated data(real and imaginary) into the copula space, i.e., $(u_1, u_2) = (F_1(x_1), F_2(x_2))$.

With the increase of r_I, tail dependence decreases, and the interference vector is becoming less sub-Gaussian but more "independent Gaussian".

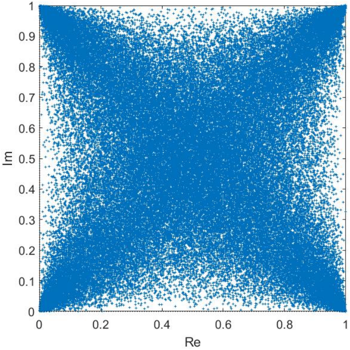


Figure 8: Copula space with $r_I = 0$

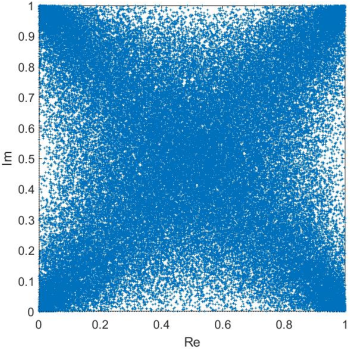


Figure 9: Copula space with $r_I = 5$

Numerical Results

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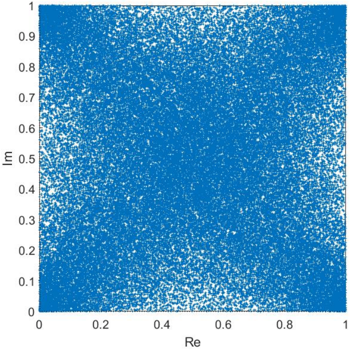

Figure 10: Copula space with $r_I = 10$

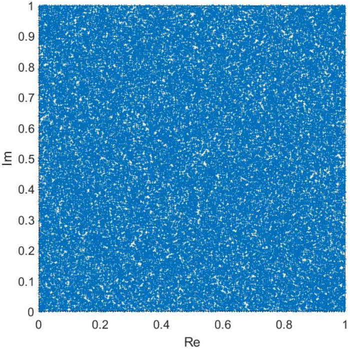

Figure 11: Copula space with $r_I = 50$

Section 3

System Model

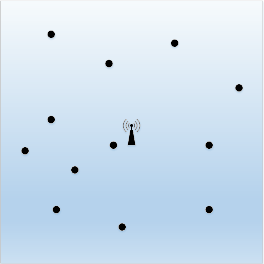

Figure 12: System Model

- Each interfering device transmits over a subset of orthogonal frequency bands, $\mathcal{B} = \{1, 2, \ldots, K\}$

System Model

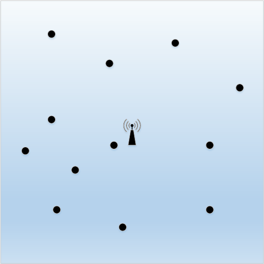

Figure 12: System Model

- Each interfering device transmits over a subset of orthogonal frequency bands, $\mathcal{B} = \{1, 2, \ldots, K\}$
- At each time t, interfering devices independently transmit on a band $k \in \mathcal{B}$, with probability $p > 0$

System Model

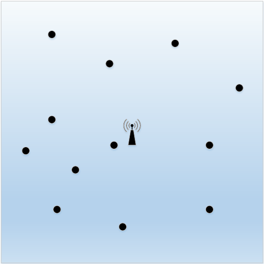

Figure 12: System Model

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- At each time t, interfering devices independently transmit on a band $k \in \mathcal{B}$, with probability $p > 0$
- On M out of K bands — $\binom{K}{M} p^M (1-p)^{K-M}$

System Model

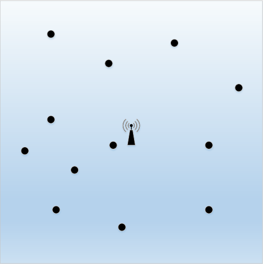

Figure 12: System Model

- Each interfering device transmits over a subset of orthogonal frequency bands, $\mathcal{B} = \{1, 2, \ldots, K\}$
- At each time t, interfering devices independently transmit on a band $k \in \mathcal{B}$, with probability $p > 0$
- On M out of K bands — $\binom{K}{M} p^M (1-p)^{K-M}$
- Interfering nodes: PPP with $p\lambda$ on the whole plane — Φ

p – proportional to the quantity of data each interfering device seeks to transmit

Dependence – interference on two bands have partial common interfering nodes.

Interference Model

Interference on band k is

$$Z_k = \sum_{j \in \Phi_k} r_{j,k}^{-\eta/2} h_{j,k} x_{j,k}, \tag{8}$$

The interference vector:

$$\mathbf{Z} = [\mathrm{Re}(z_1), \mathrm{Im}(z_1), \ldots, \mathrm{Re}(z_K), \mathrm{Im}(z_K)]^T. \tag{9}$$

Remark 1

*When p is sufficiently small, interference on each band can be approximated as independent. $\mathbf{Z}$ is composed of K independent sub-Gaussian pairs (**2-dimensional sub-Gaussian**).*

Interference Model

Interference on band k is

$$Z_k = \sum_{j \in \Phi_k} r_{j,k}^{-\eta/2} h_{j,k} x_{j,k}, \tag{8}$$

The interference vector:

$$\mathbf{Z} = [\mathrm{Re}(z_1), \mathrm{Im}(z_1), \ldots, \mathrm{Re}(z_K), \mathrm{Im}(z_K)]^T. \tag{9}$$

Theorem 4

Suppose that $h_{j,k} x_{j,k}$ is an isotropic complex random variable for $k = 1, \cdots, K$, and

$$\mathbb{E}[|\mathrm{Re}(h_{j,k} x_{j,k})|^{4/\eta}|] < \infty, \quad k = 1, \cdots, K \tag{10}$$

$\mathbf{Z}$ *follows sub-Gaussian distribution with $\alpha = 4/\eta$, when $p = 1$,*

Good model for $0 < p < 1$?

Copula Theory

Definition 5 (Sklar's Theorem)

The joint distribution function of a random vector in $\mathbb{R}^n$ $\mathbf{X} = [X_1, \dots, X_n]$ can be given in the form

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)), \tag{11}$$

where $C : [0,1]^n \to [0,1]$ is called a copula function, and $F_i,\ i = 1, \dots, n$ are the marginal distribution functions.

Remark 1

The probability density function is

$$p_{\mathbf{X}}(x_1, \dots, x_n) = c(F_1(x_1), \dots, F_n(x_n)) \prod_{i=1}^{n} p_{X_i}(x_i). \tag{12}$$

$c : [0,1]^n \to \mathbb{R}_+$ *captures dependence structure.*

T Copula

Definition 6

The t copula is defined as

$$C^t_{v,\Sigma}(\mathbf{u}) = F_{v,\Sigma}(F_v^{-1}(u_1), \ldots, F_v^{-1}(u_n))). \tag{13}$$

where $F_v(x)$ is the CDF of t distribution at x, and $F_{v,\Sigma}(\mathbf{x})$ is the joint CDF of multivariate t distribution at $\mathbf{x}$.

- Flexible dependence structure — v,Σ,
- Tractability — computationally feasible.

Numerical Results

Definition 7 (KL Divergence)

For distributions P and Q of continuous random variable, the Kullback–Leibler divergence is defined as

$$D_{KL}(P||Q) = \int_{-\infty}^{\infty} p(x) log \left(\frac{p(x)}{q(x)} \right) \mathrm{d}x. \tag{14}$$

1. Fitting the model to a data set obtained via Monte Carlo simulations,
2. Compare the KL-divergence between the fitted model and the simulated data set.

Three Models

1. T copula;
2. 2-**dim. sub-Gaussian model**: consisting of independent two-dimensional sub-Gaussian random vectors;
3. $2K$ **sub-Gaussian model**: $2K$-dimensional sub-Gaussian vector.

Numerical Results

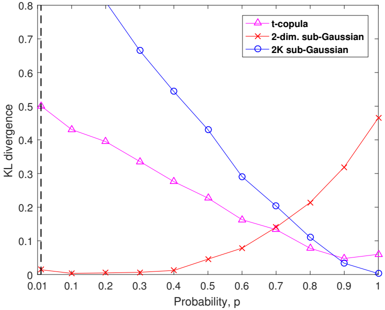

Figure 13: KL divergence between the simulated data set and three statistical models, $K = 2$

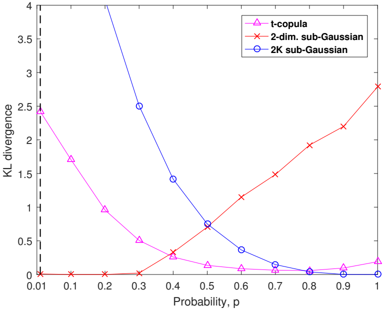

Figure 14: KL divergence between the simulated data set and three statistical models, $K = 8$

1. 2-dim. sub-Gaussian model – low D_{KL} for $p \approx 0$
2. $2K$ sub-Gaussian model – low D_{KL} for $p \approx 1$.
3. For $p > 0.7$ ($K = 2$) and $p > 0.4$ ($K = 8$), the t-copula model performs better.
4. Adopt the 2-dim. sub-Gaussian model for small p and the t-copula model for larger p

Section 4

Tail Dependence

Definition 8 (Pearson Correlation Coefficient)

Given a pair of random variables (X, Y), the Pearson coefficient is defined as

$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} \tag{15}$$

For α-stable marginals, $\rho_{X,Y}$ does not exist.

Tail Dependence

Definition 9

The lower tail dependence is defined as:

$$\lambda_l = \lim_{u \to 0} P\left(X_2 < F_2^{-1}(u) | X_1 < F_1^{-1}(u)\right) \tag{16}$$

The upper tail dependence is defined as:

$$\lambda_u = \lim_{u \to 1} P\left(X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)\right) \tag{17}$$

Conditional outage probability. Impulsive noise or interference that appears in one slot given that it exist in another slot.

Tail Dependence

Physical Meaning

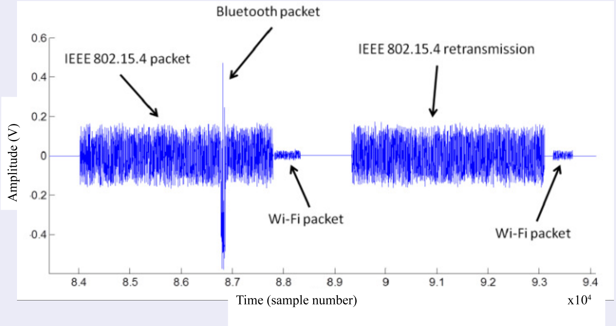

Figure 15: Coexistence of technologies in the 2:4-GHz band. Measurements made by a National Instruments USRP

Impulsive noise or interference that appears in one slot given that it exist in another slot.

Tail Dependence

The tail dependence can be used for the estimation of parameters of t copula.

Property 1

For t copula $C_{v,\Sigma}$, tail dependence can be expressed as

$$\lambda := \lambda_L = \lambda_U = 2\bar{t}_{v+1}\left(\sqrt{\frac{(1+v)(1-\rho)}{1+\rho}}\right), \tag{16}$$

where $\bar{t}_{v+1}(x) = 1 - t_{v+1}(x)$ and t_{v+1} is the Student distribution function with $v+1$ degrees of freedom.

Numerical Results

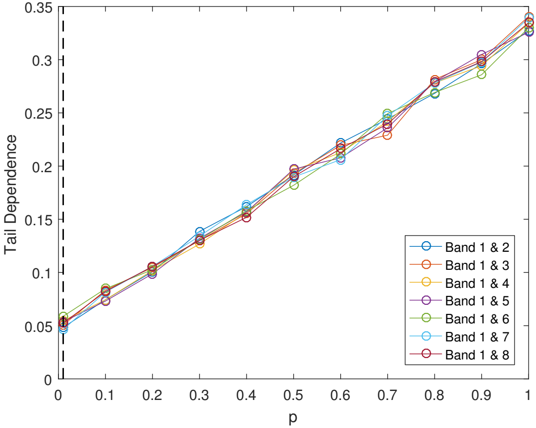

Figure 16: Estimated Tail Dependence for Real Components

Section 5

Conclusion

- One journal paper accepetd in EURASIP: Egan, Malcolm, Laurent Clavier, **Ce Zheng**, Mauro De Freitas, and Jean-Marie Gorce. "*Dynamic interference for uplink SCMA in large-scale wireless networks without coordination*." EURASIP Journal on Wireless Communications and Networking 2018, no. 1 (2018): 213.
- One paper accepted in ICC: **Ce Zheng**, Malcolm Egan, Laurent Clavier, Gareth W. Peters and Jean-Marie Gorce, "*Copula-Based Interference Models for IoT Wireless Networks*", 2019 IEEE International Conference on Communications (ICC)
- One journal paper preparing: "Statistical Characterization, Estimation and Simulation of Interference Random Vectors in Poisson Spatial Fields of Interferers"
- One conference paper going to submit to GRETSI: "On the Validity of Isotropic Complex α-Stable Interference Models for Interference in the IoT"