Copula Theory and Dependence in Interference

Chris (ZHENG Ce)

IEMN-IRCICA, CNRS

University of Lille 1

Ce.zheng@etudiant.univ-lille1.fr

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1. Copula Theory
2. Measures of Dependence
3. Two Families of Copula
4. Observation

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Copula Theory

Question : What is copula?

1. Preliminaries:

X, Y are random variables with distribution functions F(X) and G(Y), respectively. And their joint distribution function is H(X,Y). If X and Y are independent, there is a relationship:

$$H(X,Y)=F(X)G(Y) \tag{1-1}$$

What if X and Y are not independent?

Copula Theory

Question : What is copula?

2. Sklar's Theorem:

If H(X,Y) is joint distributed function with margins F(X), G(Y), there exists a (Copula) funcion C such that

$$H(X,Y) = C\big(F(X), G(Y)\big) \quad (1\text{-}2)$$

where C is unique if F and G are continuous, otherwise C is uniquely determined on $\mathrm{Ran}(F) \times \mathrm{Ran}(G)$.

Copula Theory

Question : What is copula?

2. Sklar's Theorem:

$$H(X,Y)=C(F(X),G(Y)) \tag{1-2}$$

Let u=F(x), v=G(y). C(u,v) is called copula funcion.

$$C(u,v)=H(F^{-1}(u),G^{-1}(v)) \tag{1-3}$$

$$C(u,v)=P(U\le u,V\le v) \tag{1-4}$$

C(u,v) is a distributed function with uniform margins.

Copula Theory

Question : What is copula?

3. Multivariate Copula:

$$H\left(X_1,\ldots,X_n\right)=C\left(F_1\left(X_1\right),\ldots,F_n\left(X_n\right)\right) \quad (1\text{-}5)$$

$$C\left(u_1,\ldots,u_n\right)=H\left(F_1^{-1}\left(u_1\right),F_n^{-1}\left(u_n\right)\right) \quad (1\text{-}6)$$

where $u_1 = F_1(X_1), \ldots, u_n = F_n(X_n)$

Copulas join or couple multivariate distribution functions to their one-dimension marginal distribution functions[1] .

[1] Nelsen, R. B. (2007). *An introduction to copulas*. Springer Science & Business Media.

Copula Theory

Question : Relation between copula and our research

4. Receiver:

In a SIMO system, the received signal is:

$$Y = S + I \quad \text{(1-7)}$$

where S is a vector containing the repeated sample s, and $\mathrm{I} = (i_1, \ldots i_n)$ is the interference vector.

Log likelihood ratio (**LLR**) will be:

$$\wedge(Y) = \log \frac{P\left(y_1 = s+i_1, \ldots, y_n = s+i_n \mid s = +1\right)}{P\left(y_1 = s+i_1, \ldots, y_n = s+i_n \mid s = -1\right)}$$

$$= \log \frac{h\left(y_1 - 1, \ldots, y_n - 1\right)}{h\left(y_1 + 1, \ldots, y_n + 1\right)} \quad \text{(1-8)}$$

where h is the joint **PDF** of I.

Copula Theory

Question : Relation between copula and our research

4. Receiver:

Recall that $H(X_1, \dots, X_n) = C(F_1(X_1), \dots, F_n(X_n))$. We have

$$h(\mathrm{I}) = \underbrace{c\left(F\left(i_1\right), \dots, F\left(i_n\right)\right)}_{\text{Dependent component}} \underbrace{\prod_{k=1}^{n} f\left(i_k\right)}_{\text{Independent component}} \tag{1-9}$$

where c is the density of copula:

$$c\left(u_1, \dots, u_n\right) = \frac{\partial^n C\left(u_1, \dots, u_n\right)}{\partial u_1 \dots \partial u_n} \tag{1-10}$$

Copula Theory

Question : Relation between copula and our research

4. Receiver:

$$h(\mathrm{I}) = c\left(F(i_1),\ldots,F(i_n)\right)\prod_{k=1}^{n} f(i_k) \qquad (1\text{-}9)$$

Dependent strcuture

Independent structure

$$\wedge(Y) = \log\frac{h(y_1-1,\ldots,y_n-1)}{h(y_1+1,\ldots,y_n+1)} \qquad (1\text{-}8)$$

Combining equation (1-9) and (1-8), the LLR becomes:

Copula Theory

Question : Relation between copula and our research

4. Receiver:

Combining equation (1-9) and (1-8), the LLR becomes:

$$\wedge(Y) = \underbrace{\log \frac{c\left(F\left(y_1 - 1\right), \ldots, F\left(y_n - 1\right)\right)}{c\left(F\left(y_1 + 1\right), \ldots, F\left(y_n + 1\right)\right)}}_{\wedge_c(Y)} + \underbrace{\sum_{k=1}^{n} \log \frac{f\left(y_k - 1\right)}{f\left(y_k + 1\right)}}_{\wedge_{\perp}(Y)} \qquad (1\text{-}11)$$

$\wedge_c(Y)$: Dependent strcuture → tricky

$\wedge_{\perp}(Y)$: Independent structure

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Measures of Dependece

1. Linear Dependence:

Pearson's Correlation Coefficient:

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]}\sqrt{Var[Y]}} \qquad (2\text{-}1)$$

$$\rho[\alpha_i + \beta_i X, \alpha_j + \beta_j X] = \rho[X,Y], \quad \beta_i, \ \beta_j > 0 \qquad (2\text{-}2)$$

1. Invariant under increasing linear transformation
2. Not suitable for alpha-stable, no variance ($\alpha<2$).

Measures of Dependece

2. Quadrant Dependence:

If X,Y are independent, then

$$C\left(F(X),G(Y)\right)=H(X,Y)=F(X)G(Y) \tag{2-3}$$

$$C(u,v)=uv \tag{2-4}$$

We call $C(u_1,\ldots,u_n)=\prod_{i=1}^{n}u_i$ independent copula.

Measures of Dependece

2. Quadrant Dependence:

We call X,Y are positively quadrant dependent (**PQD**) if

$$P(X \le x, Y \le y) \ge P(X \le x) P(Y \le y) \tag{2-5}$$

$$C(u,v) \ge uv \tag{2-6}$$

we can prove equivalently :

$$P(X \ge x, Y \ge y) \ge P(X \ge x) P(Y \ge y) \tag{2-7}$$

Analogously, we can define the **NQD** by reversing the sense of inequality :

Measures of Dependece

2. Quadrant Dependence:

Positively Quadrant Dependent (**PQD**):

$$P\left(X \le x, Y \le y\right) \ge P\left(X \le x\right) P\left(Y \le y\right) \qquad (2\text{-}5)$$

$$P\left(Y \le y | X \le x\right) \ge P\left(Y \le y\right) \qquad (2\text{-}8)$$

Y is more likely to have small values if the value of X is small

$$P\left(Y \le y | X \le x\right) \ge P\left(Y \le y | X \le \infty\right) \qquad (2\text{-}9)$$

A stronger condition: $P(Y \le y | X \le x)$ is nonincreasing.

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Measures of Dependece

3. Tail Dependence:

Mainly interested in the dependence among extremal values, we define lower and upper tail dependence[2]:

$$\lambda_l = \lim_{u \to 0} P\left[X \le F^{-1}(u) \mid Y \le F^{-1}(u) \right] = \lim_{u \to 0} \frac{C(u,u)}{u} \qquad (2\text{-}10)$$

$$\lambda_u = \lim_{u \to 1} P\left[X \ge F^{-1}(u) \mid Y \ge F^{-1}(u) \right] = \lim_{u \to 1} \frac{1 - 2u + C(1-u, 1-u)}{1-u} \qquad (2\text{-}11)$$

[2] "Skew-t copula for dependence modelling of impulsive(α-stable) interference." *Communications (ICC), 2015 IEEE International Conference on*. IEEE, 2015..

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Two Families of Copula

1. Archimedean Familiy:

The Archimedean copula has the form:

$$C(u_1,\ldots,u_n)=\varphi^{-1}(\varphi(u_1)+\ldots+\varphi(u_n)) \quad (3\text{-}1)$$

where φ is the generator of copula and it is a continuous, strictly decreasing function from [0,1] to [0,∞] and $\varphi(1)=0$. φ^{-1} is the inverse of φ.

According to [1], as long as φ^{-1} is completely monotonic, C is a copula. For bivariate case, φ^{-1} is convex.

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Two Families of Copula

1. Archimedean Familiy:

(1) Clayton Copula:

$$C_\theta(u_1,\ldots,u_n)=\max\left(\left[u_1^{-\theta}+\ldots+u_n^{-\theta}-(n-1)\right]^{-1/\theta},0\right) \qquad (3\text{-}2)$$

Bivariate case:

$$C_\theta(u,v)=\max\left(\left[u^{-\theta}+v^{-\theta}-1\right]^{-1/\theta},0\right) \qquad (3\text{-}3)$$

Generator:

$$\varphi(t)=\frac{t^{-\theta}-1}{\theta},\qquad \theta>0 \qquad (3\text{-}4)$$

Two Families of Copula

1. Archimedean Familiy:

(2) Gumbel Copula:

$$C_\theta\left(u_1,\ldots,u_n\right)=\exp\left\{-\left[\left(-\ln u_1\right)^\theta+\ldots+\left(-\ln u_n\right)^\theta\right]^{-1/\theta}\right\} \tag{3-5}$$

Bivariate case:

$$C_\theta\left(u,v\right)=\exp\left\{-\left[\left(-\ln u\right)^\theta+\left(-\ln v\right)^\theta\right]^{-1/\theta}\right\} \tag{3-6}$$

Generator:

$$\varphi\left(t\right)=\left(-\ln t\right)^\theta,\quad \theta\geq 1 \tag{3-7}$$

Two Families of Copula

1. Archimedean Familiy:

Tail dependence:

Clayton:

$$\lambda_l = 2^{-1/\theta}, \qquad \lambda_u = 0 \qquad (3\text{-}8)$$

Gumbel:

$$\lambda_l = 0, \qquad \lambda_u = 2 - 2^{1/\theta} \qquad (3\text{-}9)$$

Two Families of Copula

1. Archimedean Familiy:

Simulations:

1. assume Cauchy Interference
2. Clayton Copula or Gumbel Copula or independent

Figure 1. Claton Copula

Two Families of Copula

1. Archimedean Familiy:

Simulations:

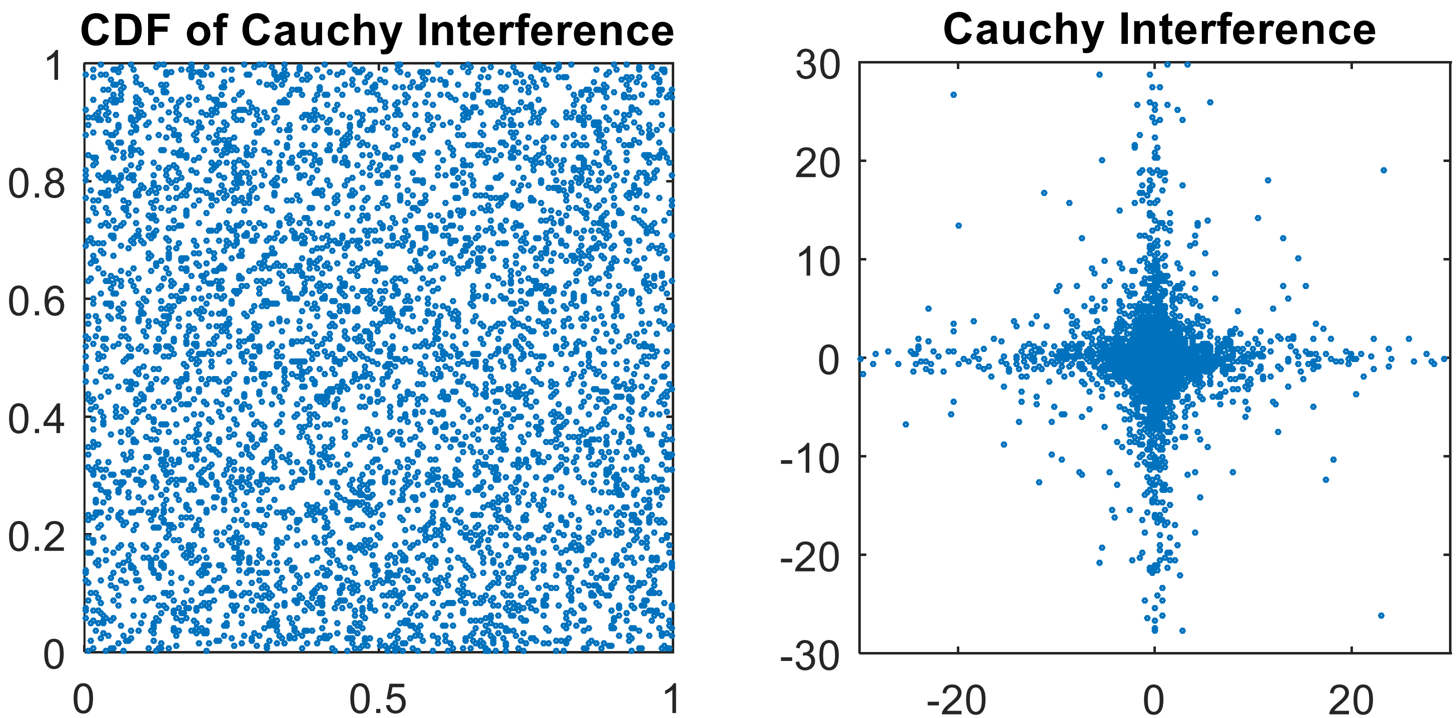


Figure 1. Cauchy Interference with Indepedence

Figure 1. Clayton Copula

Two Families of Copula

1. Archimedean Familiy:

Simulations:

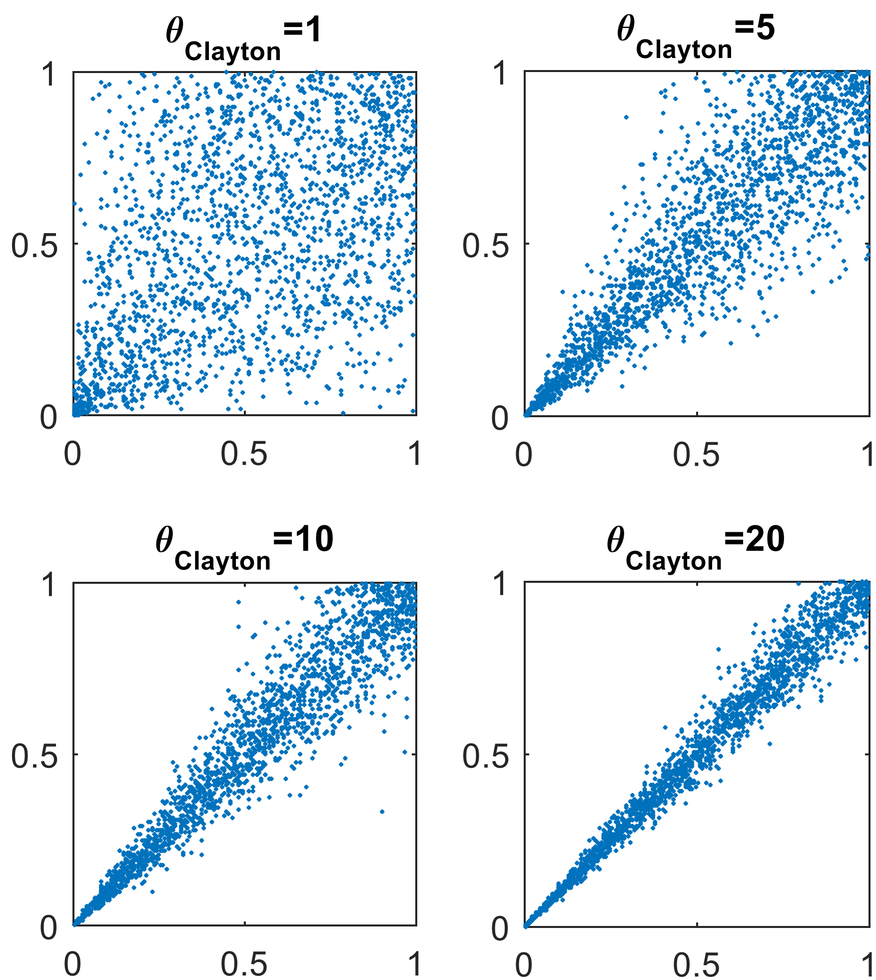


Figure 2. Clayton samples

Clayton:

$$\lambda_l = 2^{-1/\theta}, \qquad \lambda_u = 0$$

1. Zero upper tail dependence
2. With the increase of θ, it is more dependent.

Two Families of Copula

1. Archimedean Familiy:

Simulations:

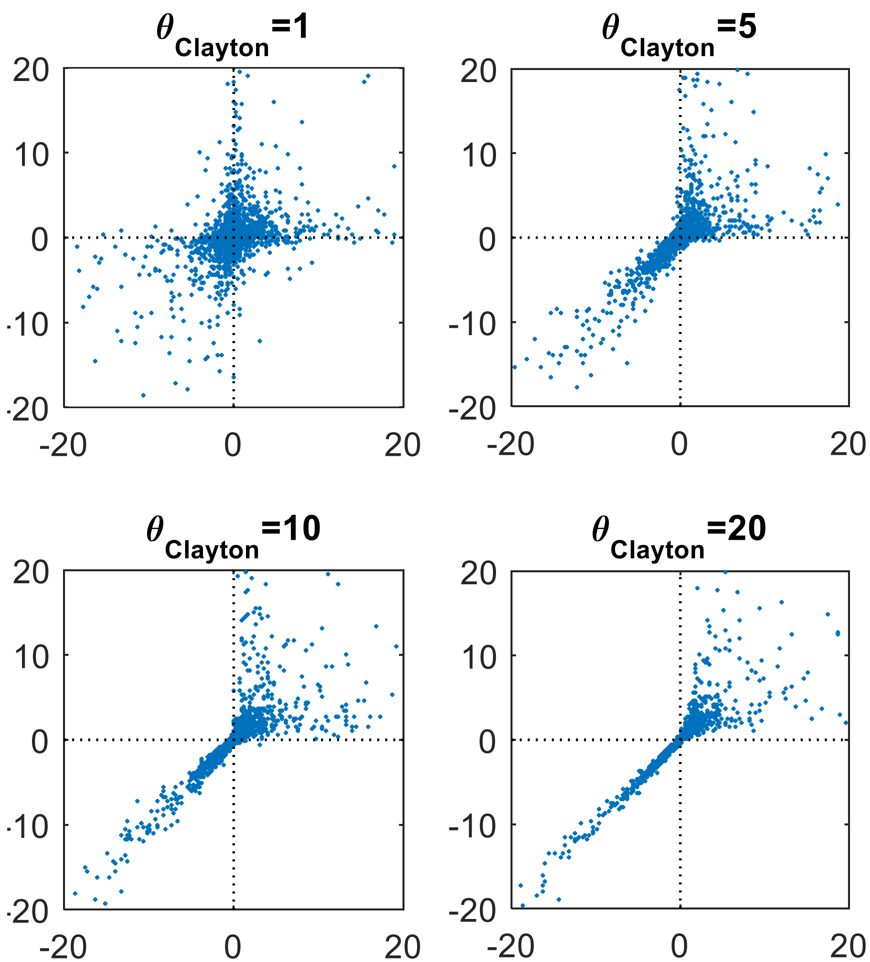


Figure 3. Cauchy Interference with Clayton

Clayton:

$$\lambda_l = 2^{-1/\theta}, \qquad \lambda_u = 0$$

1. Zero upper tail dependence
2. With the increase of θ, it is more dependent.

Two Families of Copula

1. Archimedean Familiy:

Simulations:

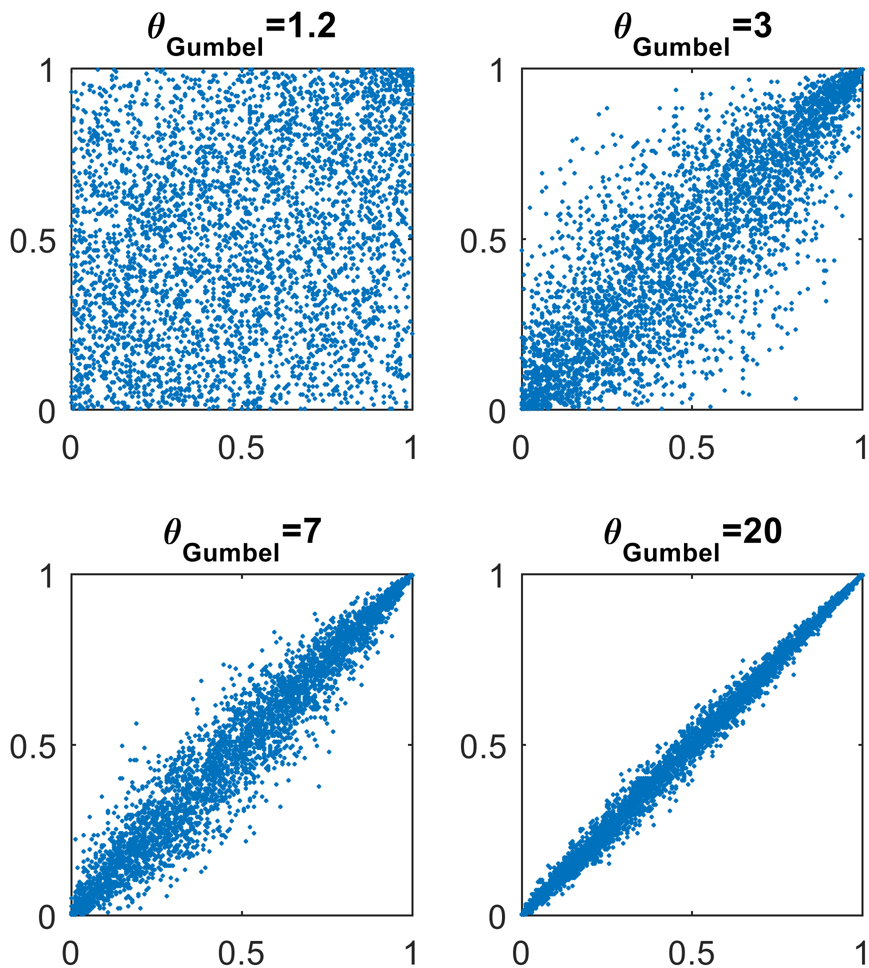


Figure 4. Gumbel samples

Gumbel:

$$\lambda_l = 0, \qquad \lambda_u = 2 - 2^{1/\theta}$$

1. Zero lower tail dependence
2. With the increase of θ, it is more dependent.

Two Families of Copula

1. Archimedean Familiy:

Simulations:

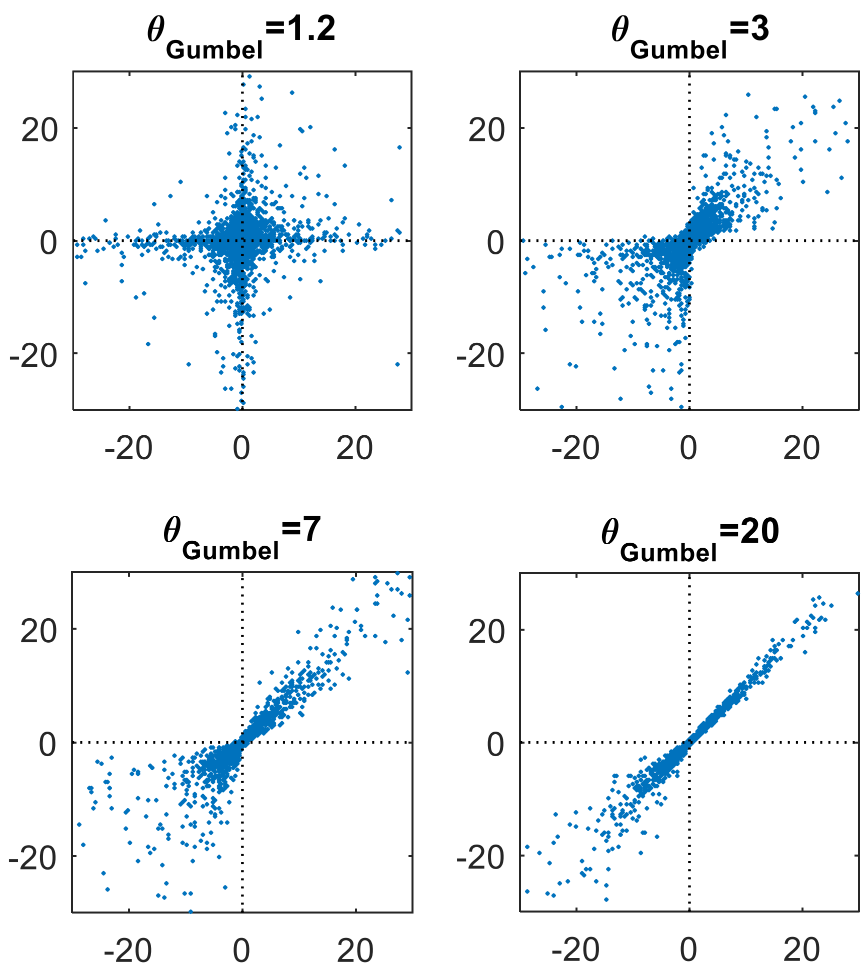


Figure 5. Cauchy Interference with Gumbel

Gumbel:

$$\lambda_l = 0, \qquad \lambda_u = 2 - 2^{1/\theta}$$

1. Zero lower tail dependence
2. With the increase of θ, it is more dependent.

Two Families of Copula

2. Elliptical Familiy:

Elliptical distribution:

$$f_{\vec{X}}\left(\vec{x}\right)=\frac{c_n}{\sqrt{\left|\Sigma\right|}}g_n\left[\frac{1}{2}\left(\vec{x}-\vec{\mu}\right)^T\Sigma^{-1}\left(\vec{x}-\vec{\mu}\right)\right] \qquad (3\text{-}10)$$

where $g_n(\cdot)$ is the density generator and c_n is the normalizing constant:

$$c_n=\frac{\Gamma\left(n/2\right)}{\left(2\pi\right)^{n/2}}\left[\int_0^\infty x^{n/2-1}g_n\left(x\right)dx\right]^{-1} \qquad (3\text{-}11)$$

Gaussian and t distribution belong to the Elliptical distribtuion.

Two Families of Copula

2. Elliptical Familiy:

(1) Gaussian Copula:

Gaussian distribution $\vec{X} \sim N(\vec{\mu}, \Sigma)$ has the density:

$$f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right] \tag{3-12}$$

Hence:

$$C(u_1, \ldots, u_n) = F_{\Sigma}^{n}\left(F^{-1}(u_1), \ldots, F^{-1}(u_n)\right) \tag{3-13}$$

where F_{Σ}^{n} is the CDF and F is the margin.

zero tail dependence

Two Families of Copula

2. Elliptical Familiy:

(2) t Copula:

t distribution $\vec{X} \sim t_v(\vec{\mu}, \Sigma)$ has the density:

$$f_{\vec{X}}(\vec{x}) = \frac{\Gamma\left[(v+n)/2\right]}{\Gamma\left[v/2\right] v^{n/2} \pi^{n/2} \left|\Sigma\right|^{1/2}} \exp\left[1 + \frac{1}{v}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right] \qquad (3\text{-}12)$$

where v is the degree of freedom and n is the dimension:

$$C(u_1, \ldots, u_n) = t_{v,\Sigma}^n\left(t_v^{-1}(u_1), \ldots, t_v^{-1}(u_n)\right) \qquad (3\text{-}13)$$

where $t_{v,\Sigma}^n$ is the CDF and t_v^{-1} is the margin.

symmetric tail dependence.

Two Families of Copula

2. Elliptical Familiy:

Simulations:

Gaussian copula:

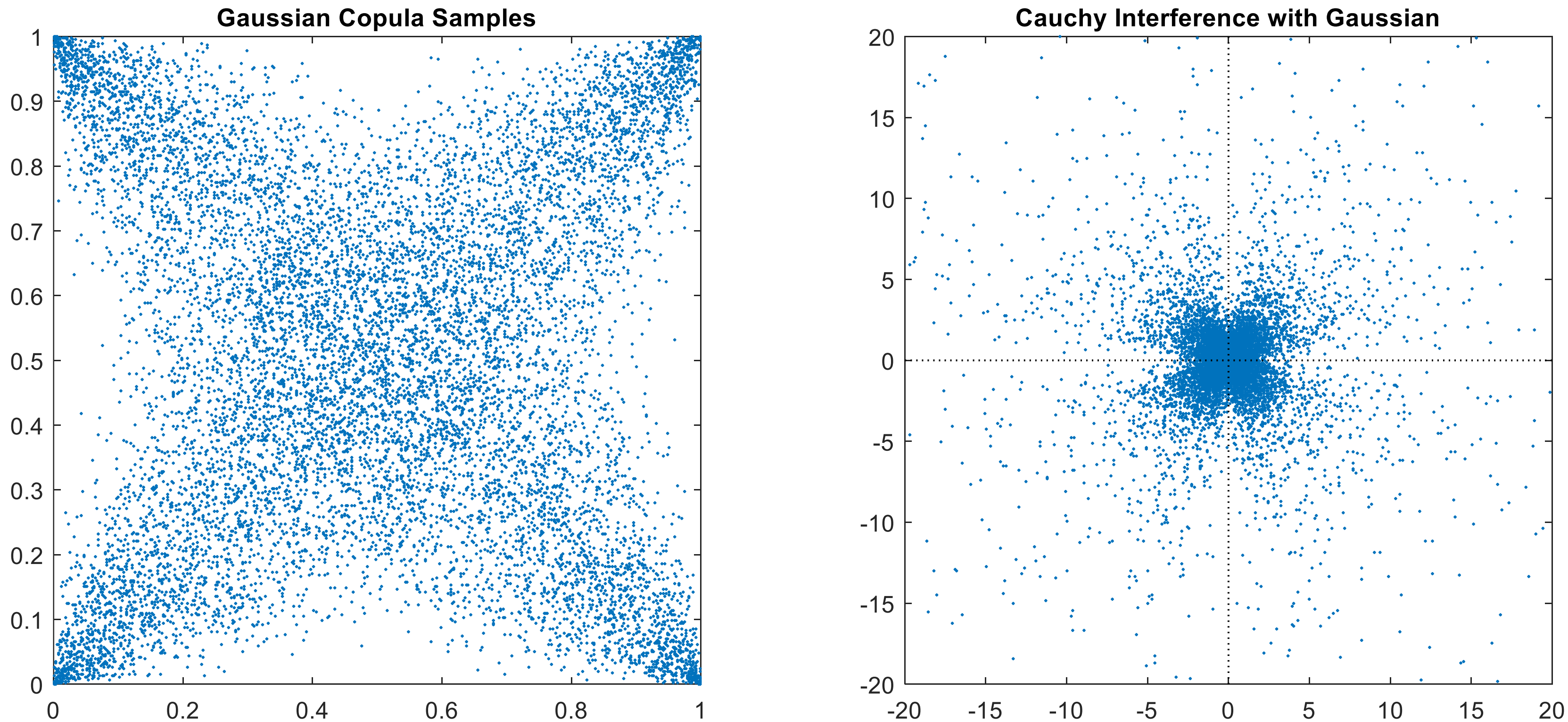


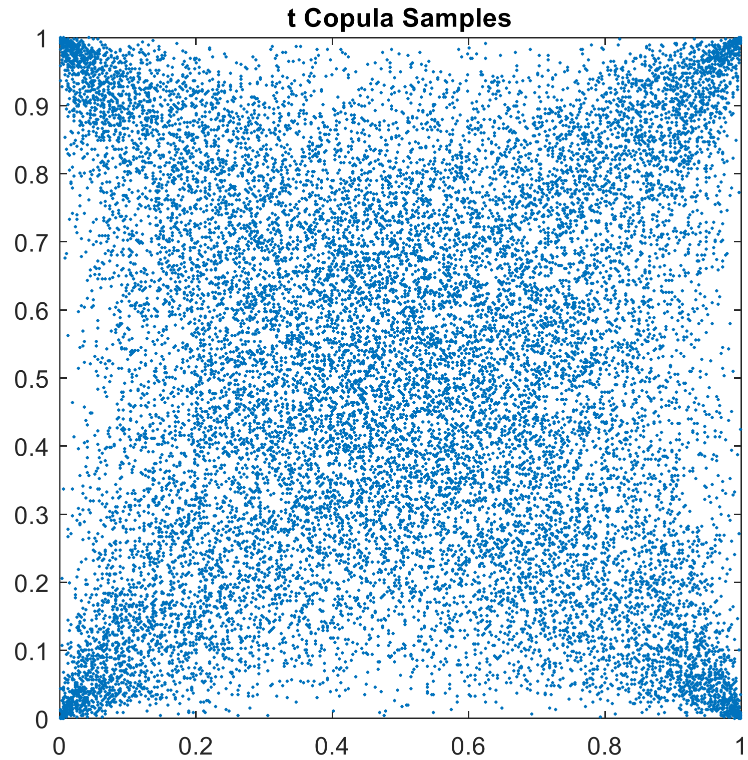
Figure 6. Gaussian copula samples and interference

Two Families of Copula

2. Elliptical Familiy:

Simulations:

t copula:



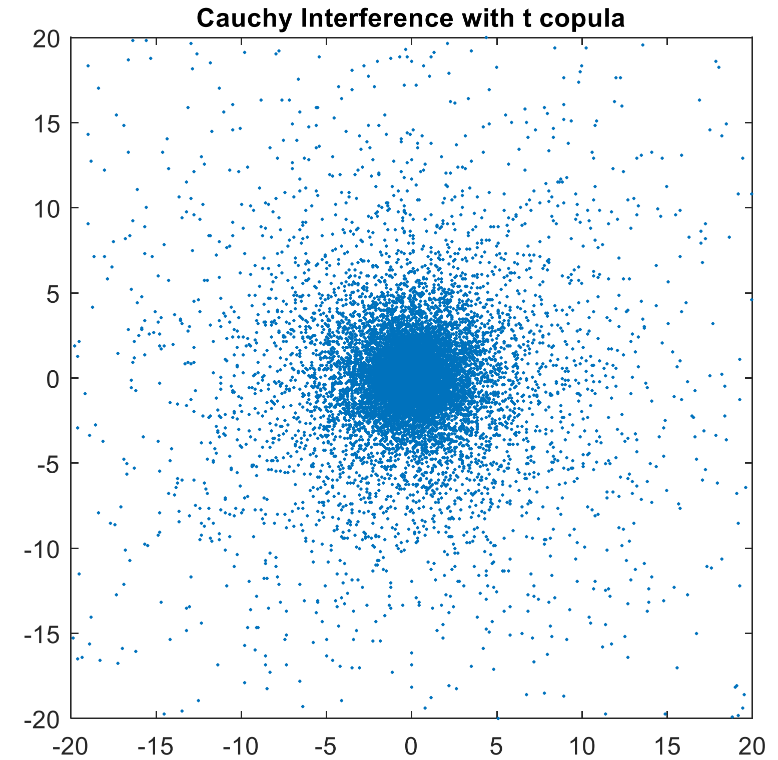


Figure 7. *t* copula samples and interference

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Observation

1. Complex Interference:

The interference is composed of real and imaginary parts:

$$Z_{2K} = (z_{1,r}, z_{1,i}, \ldots, z_{K,r}, z_{K,i}) \quad (4\text{-}1)$$

Joint distribution of Z_{2K} can be expressed as:

$$F_{Z_{2K}}(z_{1,r}, z_{1,i}, \ldots, z_{K,r}, z_{K,i}) = C\left(F\left(z_{1,r}\right), F\left(z_{1,i}\right), \ldots, F\left(z_{1,r}\right), F\left(z_{K,i}\right)\right) \quad (4\text{-}2)$$

In general, the dependence of the pair $\left(F\left(z_{k,r}\right), F\left(z_{k,j}\right)\right)$ is different with the pair $\left(F\left(z_{k,r}\right), F\left(z_{j,r}\right)\right)$ or $\left(F\left(z_{k,r}\right), F\left(z_{k,j}\right)\right)$.

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Observation

1. Complex Interference:

The Archimedean copula has the form:

$$C\left(u_1,\ldots,u_n\right)=\varphi^{-1}\left(\varphi\left(u_1\right)+\ldots+\varphi\left(u_n\right)\right) \qquad (3\text{-}1)$$

$$F_{Z_{2K}}(z_{1,r},z_{1,i},\ldots,z_{K,r},z_{K,i})=C\left(F\left(z_{1,r}\right),F\left(z_{1,i}\right),\ldots,F\left(z_{1,r}\right),F\left(z_{K,i}\right)\right) \qquad (4\text{-}2)$$

In general, the dependence of the pair $\left(F\left(z_{k,r}\right),F\left(z_{k,j}\right)\right)$ is different with the pair $\left(F\left(z_{k,r}\right),F\left(z_{j,r}\right)\right)$ or $\left(F\left(z_{k,r}\right),F\left(z_{k,j}\right)\right)$.

Observation

2. Copula Density:

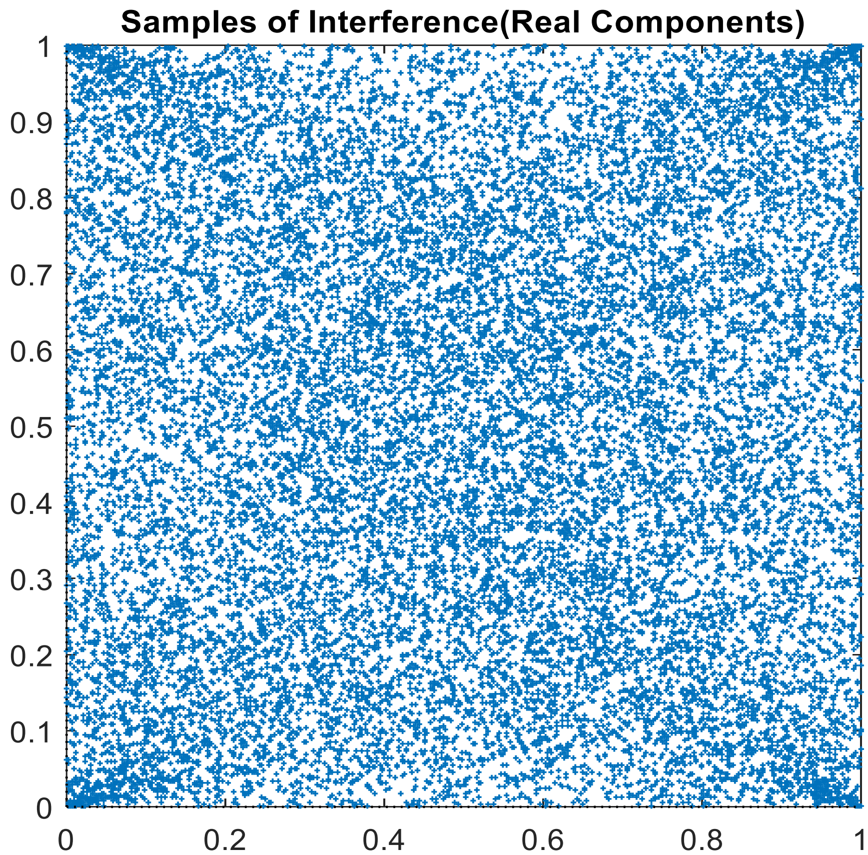

Figure 8. Samples of α-stable interference

What is the dependence structure?

Observation

2. Copula Density:

There is a dependence.

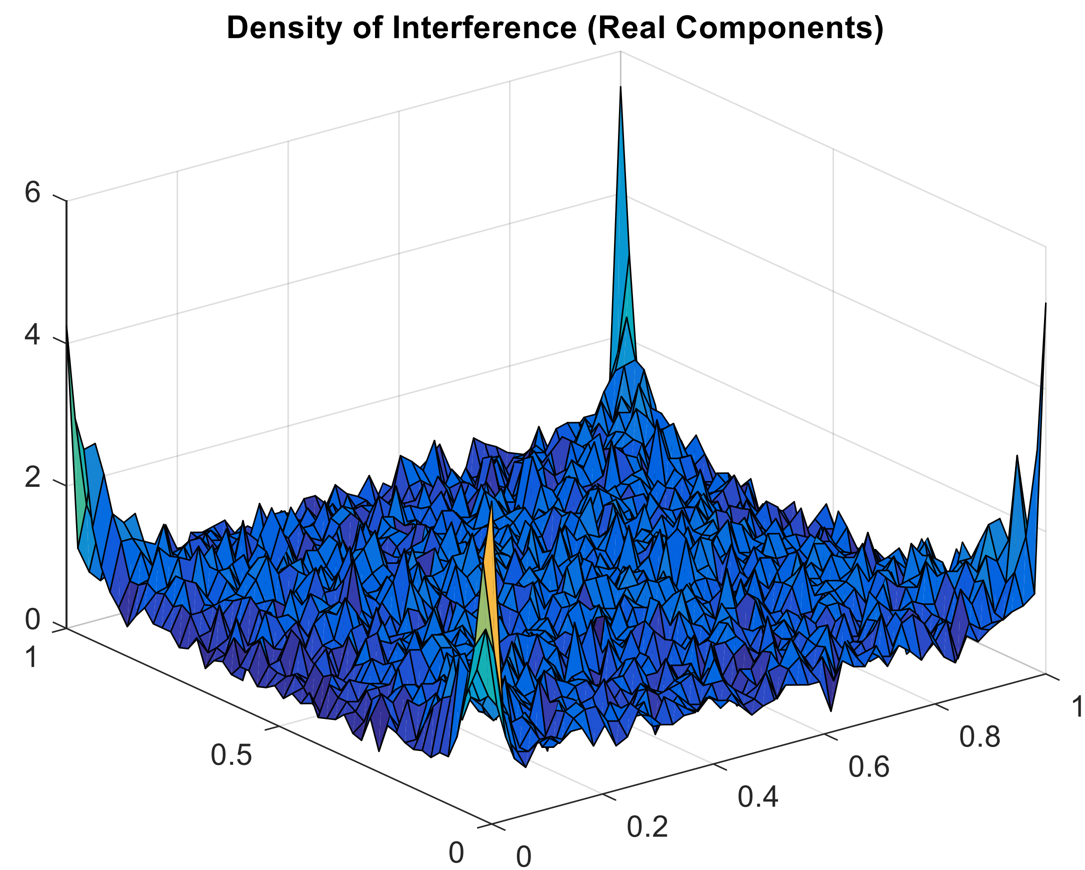

Figure 8. Density of CDF of α-stable interference

Observation

2. Copula Density:

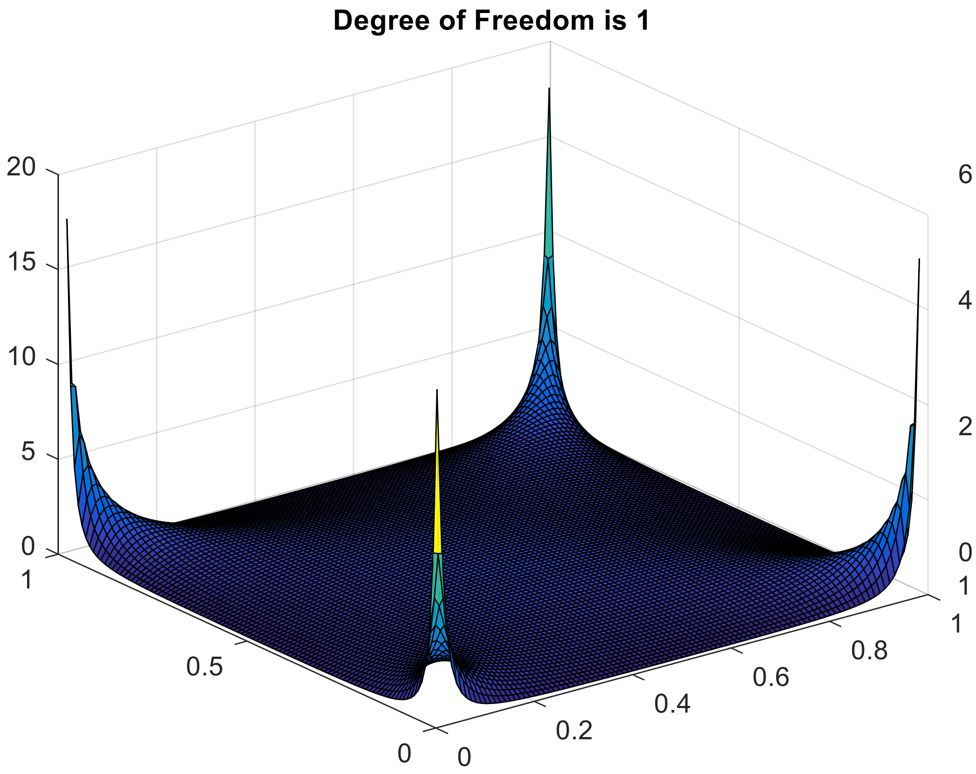

Figure 9. Density of *t* copula

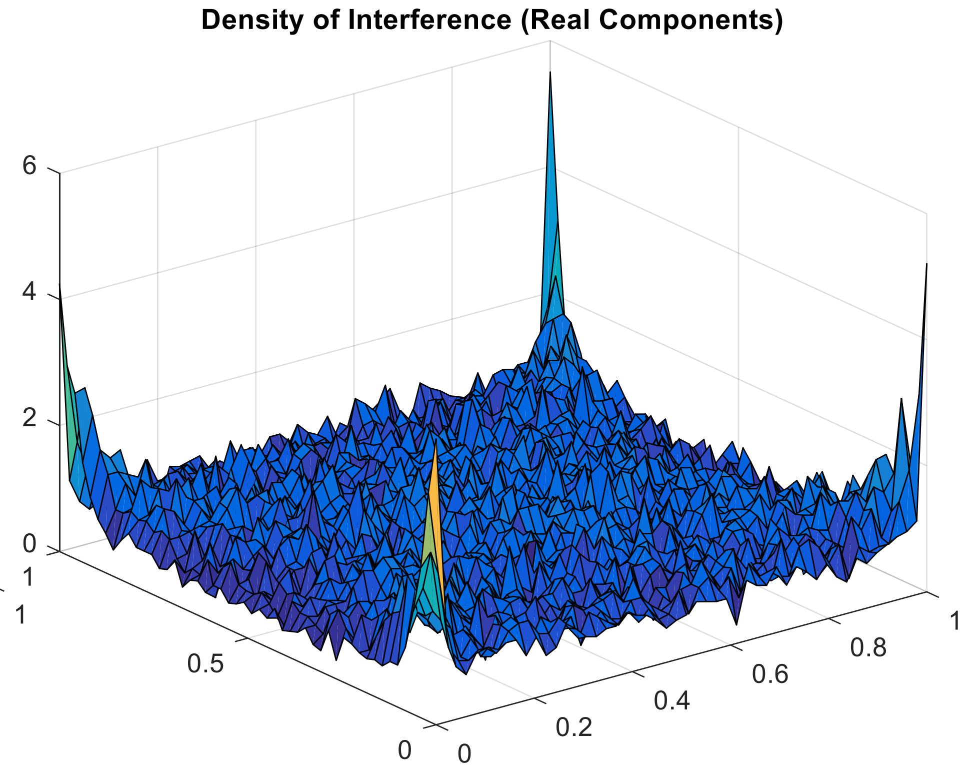

Figure 8. Density of CDF of α-stable interference

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My point of view on Copula:

1. It has a bright future since the most common assumption on the dependence is ‘independent’.

2. Hexagon vs. Stochastic Geometry

Independence vs. Copula Theory

THANK YOU !