Copula Theory in Communication Society

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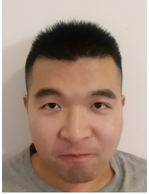
ARBURST Project



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- IRCICA, University of Lille, Lille, France
- Ph.D. 01/10/2017 – 30/09/2020

- Thesis: Impact of impulsive – dependent interference on radio communications, funded by ARBURST project and CNRS.

- Achievable Region of Multi-users Bursty Wireless Communications

Outline

Outline

Copula Theory

$X_1, \cdots, X_n$ are random variables with joint distribution funcion F and probablity density function f.

Independence: For any multivariate distribution, the distribution can be written as:

$$F(x_1, \dots, x_n) = \prod_{i=1}^{n} F(x_i),$$
$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i),$$

if $x_i, i = 1, \cdots, n$ are independent.

Copula Theory

$X_1, \cdots, X_n$ are random variables with joint distribution funcion F and probablity density function f.

Theorem 1 (Sklar's Theorem:)

For any multivariate distribution, the distribution can be written as:

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)),$$

where $C : [0,1]^n \to [0,1]$ *is called a* **Copula** *function (***Unique***).*

The probability density function is

$$f(x_1, \ldots, x_n) = \underbrace{c(F_1(x_1), \ldots, F_n(x_n))}_{\text{Dependence}} \underbrace{\prod_{i=1}^{n} f(x_i)}_{\text{Indepedence}},$$

Copula Theory

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)),$$

where

$$c(u_1, \cdots, u_n) = \frac{\partial^n C(u_1, \ldots, u_n)}{\partial u_1, \cdots, \partial u_n}$$

$$f(x_1, \ldots, x_n) = \underbrace{c(F_1(x_1), \ldots, F_n(x_n))}_{\text{Dependence}} \underbrace{\prod_{i=1}^{n} f(x_i)}_{\text{Indepedence}},$$

$$\log(f(x_1, \ldots, x_n)) = \underbrace{\log(c(F_1(x_1), \ldots, F_n(x_n)))}_{\text{Dependence}} + \underbrace{\sum_{i=1}^{n} \log(f(x_i))}_{\text{Indepedence}},$$

Remark 1

Copula enables us modeling the dependence structure separately from modeling the marginals.

Copula Theory

Theorem (Sklar's Theorem:)

For any multivariate distribution, the distribution can be written as:

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)),$$

where $C : [0,1]^n \to [0,1]$ is called a **Copula** *function (***Unique***).*

Corollary 2

Let $u_1 = F_1(x_1), \dots, u_n = F_n(x_n)$, and $x_1 = F_1^{-1}(u_1), \dots, x_n = F_n^{-1}(u_n)$

$$C(u_1, \dots, u_n) = F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)),$$

Definition 3

Let $\mathbf{U} = (u_1, \cdots, u_n)$ has uniformly distributed marginals, i.e., $u_i \sim \mathbb{U}(0,1), \ i = 1, \cdots, n$, the joint cumulative distribution is called copula function:

$$C(u_1, \dots, u_n) = P(U_1 \le u_1, \dots, U_n \le u_n),$$

Copula Theory

Remark 2

A method to construct copula:

$$C(u_1, \dots, u_n) = F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)),$$

One family of copula is the elliptical copula derived from the elliptical distributions

Two major copula families:

- Archimedean Copula – Clayton, Gumbel, etc
- Elliptical Copula – Gaussian, t-copula, etc.

Outline

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Mutual Information

$$I(\mathbf{x}) = \sum_i H(x_i) - H(\mathbf{x})$$

Theorem 4

Mutual information is the negative entropy of the corresponding copula function:

$$I(\mathbf{x}) = -H_c(\mathbf{u})$$

Definition 5 (Copula Entropy)

Let $\mathbf{x} = (x_1, \cdots, x_n)$ be the random vector with marginals $\mathbf{u} = (u_1, \cdots, u_n)$ and the copula density is $c(\mathbf{u})$. Copula entropy is defined as

$$H_c(\mathbf{u}) = -\int_{\mathbf{u}} c(\mathbf{u}) \log(c(\mathbf{u})) \mathrm{d}\mathbf{u}$$

Mutual Information

Proof.

$$
\begin{aligned}
I(\mathbf{x}) &= \int_{\mathbf{x}} p(\mathbf{x}) \log \frac{p(\mathbf{x})}{\prod_i p(x_i)} \mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{x}} c(\mathbf{u_x}) \prod_i p(x_i) \log c(\mathbf{u_x}) \mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{u_x}} c(\mathbf{u_x}) \log c(\mathbf{u_x}) \mathrm{d}\mathbf{u_x} \\
&= -H_c(\mathbf{u_x})
\end{aligned}
$$

□

$$
p(x_1, \ldots, x_n) = c(u_1, \ldots, u_n) \prod_{i=1}^{n} p(x_i),
$$

where u_i is the marginal (CDF) of x_i.

Mutual Information

$$H(\mathbf{x}) = \sum_i H(x_i) - I(\mathbf{x}) = \sum_i H(x_i) + H_c(\mathbf{u})$$

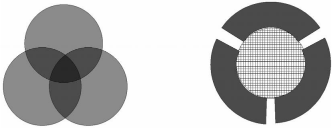

Figure 1: Two way of understanding MI.
Left: Traditional way. Right: Copula Way

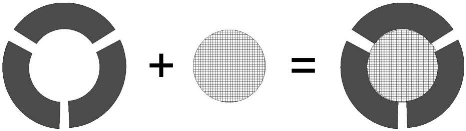

Figure 2: The relationship between the entropy of random variables, margin entropies, and the copula entropy

References

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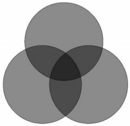
Mutual Information

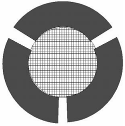
$$I(\mathbf{x}) = -H_c(\mathbf{u})$$

Remark 3

Mutual Information exactly measures the uncertainty of the dependence of random variables.

- MI used to be understood as the intersection of margin entropies. However, MI as copula entropy has no intersection with the margin entropy of each random variable





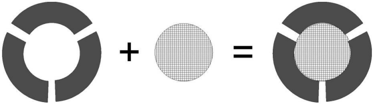
Mutual Information

$$I(\mathbf{x}) = -H_c(\mathbf{u})$$

Remark 3

Mutual Information exactly measures the uncertainty of the dependence of random variables.

- Independence between copula and margins:
  (1) Variations of random variables can have different margin entropies that have no effect on the copula entropy, i.e., MI;
  (2) Same margin entropy, but different copula functions, i.e., different dependence and different MI.



Mutual Information

Estimation of Mutual Information with copula:

$$I(\mathbf{x}) = -H_c(\mathbf{u})$$

Given a data set of samples $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ with d dimension,

- **Step 1** Determine the empirical copula density $\hat{\mathbf{U}}$ from $\mathbf{X}$

$$\hat{\mathbf{U}}_i(j) = \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_{\mathbf{x}_i(k) \leq \mathbf{x}_i(j)}$$

  where $\mathbf{x}_i(j)$ is the j-th dimension of the i-th sample $\mathbf{x}_i$
- **Step 2** Estimate entropy of $\hat{\mathbf{U}}$.

Outline

1. Copula Theory
2. Application of Copulas
   - Mutual Information
   - Blind Source Separation
   - Receiver Design
   - Diversity Combining
   - Channel Estimation
   - Copulas in Signal Processing
   - Empirical Copula
   - Other Works
3. Challenge
4. Summary & Conclusion

Blind Source Separation

Blind Source Separation (BSS) is to recover the underlying component from their mixtures, where the mixing matrix and distribution of component are unknown.

Independent Component Analysis (ICA) method:

$$\mathbf{x} = \mathbf{A}\mathbf{s} \quad \text{or} \quad \hat{\mathbf{s}} = \mathbf{W}\mathbf{x}$$
$$p(\mathbf{x}) = p(\mathbf{A}\mathbf{s}) = |\det(\mathbf{W})| p(\mathbf{s})$$
$$p(\mathbf{s}) = \prod_{i=1}^{n} p(s_i)$$

where the source signal $\mathbf{s} = [s_1, \cdots, s_n]$ is assumed to be **independent**, $\mathbf{A}$ and $\mathbf{W} = \mathbf{A}^{-1}$ is the mixing and demixing matrix to be solved.

Blind Source Separation

Copula Component Analysis (CCA) method: dependent source signals

PDF of the proposed model:

$$p_c(\mathbf{s}) = c(\mathbf{u}) \prod_{i=1}^{n} p(s_i)$$

Minimize the KL distance between the real PDF $p(\mathbf{s})$ and $p_c(\mathbf{s})$:

$$D(p||p_c) = \mathbf{E}_{p(\mathbf{s})} \log \frac{p(\mathbf{s})}{p_c(\mathbf{s})} = \mathbf{E}_{p(\mathbf{s})} \log \frac{p(\mathbf{s})}{\prod_{i=1}^{n} p(s_i)} - \mathbf{E}_{p(\mathbf{s})} \log c(\mathbf{u})$$

Blind Source Separation

$$
\begin{aligned}
D(p||p_c) &= \mathbf{E}_{p(\mathbf{s})} \log \frac{p(\mathbf{s})}{\prod_{i=1}^{n} p(s_i)} - \mathbf{E}_{p(\mathbf{s})} \log c(\mathbf{u}) \\
&= I(s_1, \cdots, s_n) + H_p(c(\mathbf{u}))
\end{aligned}
$$

The first term – KL distance between $p(\mathbf{x})$ and ICA model

The second term – entropy of copula

1. Solve $\mathbf{W}$ from $\min I(s_1, \cdots, s_n; \mathbf{W})$
2. Determine θ in $C(\mathbf{u}; \theta)$from $\max H_p(c(\mathbf{u}))$

References

3. J. Ma and Z. Sun, "Copula component analysis," in International Conference on Independent Component Analysis and Signal Separation, pp. 73–80, Springer, 2007.

4. J. Kitchen and W. Moran, "Copula techniques in wireless communica- tions," ANZIAM Journal, vol. 51, pp. 526–540, 2009.

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Receiver Design

In a **SIMO** (1×2) system, antennas close by are more likely have the large value of interference at the same time. And the received signal is

$$\begin{cases} y_1 = s + i_1 \\ y_2 = s + i_2 \end{cases}$$

where $s \in -1, 1$.

The Log Likelihood Ratio (LLR) is

$$\begin{aligned} \Lambda(y_1, y_2) &= \log \frac{f(y_1 - 1) f(y_2 - 1) c(F(y_1 - 1), F(y_2 - 1))}{f(y_1 + 1) f(y_2 + 1) c(F(y_1 + 1), F(y_2 + 1))} \\ &= \underbrace{\log \frac{f(y_1 - 1) f(y_2 - 1)}{f(y_1 + 1) f(y_2 + 1)}}_{\Lambda_{\perp}(y_1, y_2)} + \underbrace{\log \frac{c(F(y_1 - 1), F(y_2 - 1))}{c(F(y_1 + 1), F(y_2 + 1))}}_{\Lambda_c(y_1, y_2)} \end{aligned}$$

where $\Lambda_c(y_1, y_2)$ is the part of LLR depending on the copula and represents the dependence structure.

Receiver Design

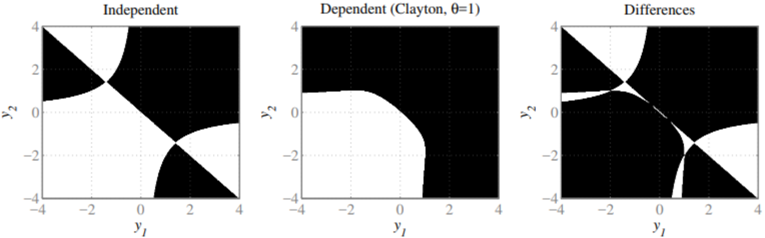

Figure 3: Decision region for independent Cauchy, Cauchy marginals and Clayton copula and the difference between both (in white the areas where the dependence structure modifies the optimal decision)

References

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Outline

Diversity Combining

Selection Combining (SC): For two channels with $SNR_i, i = 1, 2$, for independent case we have

$$\mathbf{P_{out}} = \mathbf{P}[SNR_1 \leq x, SNR_2 \leq x] = \prod_i \mathbf{P}[SNR_i < x]$$

for dependent case, we have

$$\mathbf{P_{out}} = \mathbf{P}[SNR_1 \leq x, SNR_2 \leq x] = C(F_1(x), F_2(x))$$

where $F_1(x) = \mathbf{P}[SNR_1 \leq x]$ and $F_2(x) = \mathbf{P}[SNR_2 \leq x]$.
Maximal Ratio Combining (MRC) in [8].

References

7. S. Livieratos, A. Voulkidis, G. Chatzarakis, and P. Cottis, "Correlated phenomena in wireless communications: A copula approach," in Applications of Mathematics and Informatics in Science and Engineering, pp. 95–104, Springer, 2014.
8. J. A. Ritcey, "Copula models for wireless fading and their impact on wireless diversity combining," in 2007 Conference Record of the Forty-First Asilomar Conference on Signals, Systems and Computers, pp. 1564–1567, IEEE, 2007.

Outline

Channel Estimation

1. Dependence exists between signal and noise.

The received signal is

$$r(t) = s(t) + n(t)$$

The PDF of $r(t)$ is

$$f_r(r) = \int_{-\infty}^{\infty} f_{sn}(r - n, n)dn$$

where

$$f_{sn}(s, n) = f_s(s)f_n(n)c(F_s(s), F_n(n); \theta_c)$$

References

9. M. H. Gholizadeh, H. Amindavar, and J. A. Ritcey, "Analytic nakagami fading parameter estimation in dependent noise channel using copula," EURASIP Journal on Advances in Signal Processing, vol. 2013, no. 1, p. 129, 2013.

Channel Estimation

2. Dependence exists between channels.

The received signal is

$$r(t) = q(t) + n(t) = \sum_{i=1}^{2} q_i(t) + n(t)$$

The PDF of $q(t)$ is

$$f_q(q) = \int_{-\infty}^{\infty} f_{sn}(q - q_1, q_2) dn$$

where

$$f_{q_1,q_2}(q_1, q_2) = f_{q_1}(q_1) f_{q_2}(q_2) c(F_{q_1}(q_1), F_{q_2}(q_2); \theta_c)$$

References

10. M. H. Gholizadeh, H. Amindavar, and J. A. Ritcey, "On the capacity of mimo correlated nakagami-m fading channels using copula," EURASIP Journal on Wireless Communications and Networking, vol. 2015, no. 1, p. 138, 2015.

Outline

Copulas in Signal Processing

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)),$$

Constructing Copulas for Signal Processing:

1. Log-normal copula
2. Weibull/Rayleigh/exponential copula
3. Nakagami-m copula
4. Rician copula

Methods and algorithm for generating copulas and properties of copulas

References

11. X. Zeng, J. Ren, Z. Wang, S. Marshall, and T. Durrani, "Copulas for statistical signal processing (part i): Extensions and generalization," Signal Processing, vol. 94, pp. 691–702, 2014.
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Outline

Empirical Copula

Gareth Peters showed that the non-standard dependence (tail dependence) features exist between multiple frequency bands in wireless channels in [15].

Pearson correlation coefficient only measures the linear dependence:

$$\mathbf{E}[X \cdot X^2] = 0$$

Definition 6

The lower and upper tail dependence are defined as:

$$\lambda_L = \lim_{u \to 0} P\left(X_2 \leq F_2^{-1}(u) | X_1 \leq F_1^{-1}(u)\right)$$

$$\lambda_U = \lim_{u \to 1} P\left(X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)\right)$$

Empirical Copula

Property 1

$$\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u}$$
$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u}$$

Remark 4

Tail dependence provides an approach to quantification of the dependence in extremes of a multivariate distribution and can be related directly to the parameters of the copula statistical model.

One physical scenario is the MIMO system where there is a strong interferer close to the receiver. The interference can be quite impulsive at different time slots, frequency bands and space antennas.

References

15. G. W. Peters, T. A. Myrvoll, T. Matsui, I. Nevat, and F. Septier, "Communications meets copula modeling: Non-standard dependence features in wireless fading channels," in 2014 IEEE Global Conference on Signal and Information Processing (GlobalSIP), pp. 1224–1228, IEEE, 2014.

Outline

Other Works

1. Pin-Hsun Lin and Eduard A. Jorswieck studies the relation between **copula and channel orders** [16],[17]
2. **Dealing uncertainty channel models**: Major works are done by Ezio
3. In [21], [22], the concept of **cumulative capacity** was brought up. And bounds are derived with copulas.

References

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22 F. Sun, Y. Jiang, and L. Li, "A study on further properties of wireless channel capacity," in 2016 IEEE/CIC International Conference on Communications in China (ICCC), pp. 1–6, IEEE, 2016.

Outline

Challenge

Challenge: Choosing the right copula with properties

1. Captures the characteristics of the dependence structure, such as symmetry or exchangeability;
2. The closure property under the taking of margins, i.e., the bivariate margins or higher-order margins belongs to the same parametric family;
3. Flexibility and wide range of dependence which means one can get wide-ranging dependence by varying one or several paramters;
4. Tractability, i.e., closed form of representation of marginals and densities or numerically computationally feasible to work with.

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Summary & Conclusion

- Copula serves as a powerful mathematical tool for modeling the dependence structure between random variables;

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Summary & Conclusion

- Copula serves as a powerful mathematical tool for modeling the dependence structure between random variables;
- It is now gradually applied in the academic study of communication area, and has great potential in its application;
- The most popular dependence in our previous study is "independence";
- Copula always outperfroms independence even it is independent. Choosing the right copula, "worst" case is independent.